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Atkinson, D.; Kaekebeke, M.; de Roo, M.

Published in:
Journal of Mathematical Physics

DOI:
10.1063/1.522581

Phase shifts as functions of the cross section

D. Atkinson, M. Kaekebeke, and M. de Roo

Institute for Theoretical Physics, Postbox 800 wsn 4, Groningen, The Netherlands
(Received 8 February 1974)

We show that an elastic scattering amplitude may be defined as an implicit function of the differential cross section. A practical method is given for a numerical investigation of this dependence, both in the elastic and in the inelastic regions. In particular, we consider the case of a polynomial amplitude, and we show that the Crichton ambiguity is not isolated.

1. INTRODUCTION

In recent years our understanding of the nature of the unitarity constraint upon scattering amplitudes has greatly improved.[1-32] In particular, sufficient conditions under which an elastically unitary scattering amplitude is unique, once the differential cross section is specified, have been given.[4-7] Further, the existence of a continuum ambiguity in the inelastic region has been investigated in detail.[7-17] Some preliminary investigations of spin and isospin complications have also been made.[12,13,24,25,29,30]

In this paper we investigate a different, but mathematically similar problem: given the differential cross section *and* a set of phase shifts that fit it, we show that in general one may change the cross section by a small increment, and find the correspondingly altered phase shifts that fit the new cross section. This is interesting (if not wholly unexpected) from an epistomological point of view; and it is also of importance practically, for it means that we can explore systematically the uncertainty in the phase shifts that is generated by the experimental error associated with a measurement of the differential cross section. In this paper we limit our attention for simplicity to spinless, isospinless scattering.

The basic result of the present work is set out and proved in Sec. 2. Here we use the Hildebrandt—Graves theorem to show that the unitarity equation defines the amplitude as an implicit function of the cross section. We employ a certain set of Hilbert spaces of functions that are analytic in the $\cos\theta$-plane, and the key of the proof is the demonstration that the Fréchet derivative of the unitarity mapping is compact. For the singular situation in which unity belongs to the point spectrum of this derivative, we show by means of bifurcation theory that the amplitude is still defined (though in general no longer unique). The precise nature of the solution manifold in the vicinity of a singular point depends on the number of real solutions of the bifurcation equation.

In Sec. 3 we show how to prevent zeros of the dispersive part from turning into unwanted square-root branch points, and in Sec. 4 we sketch the Newton—Kantorovich method that can be used for a numerical investigation. In Sec. 5 we generalize the method to the inelastic region, where we now have the choice of varying the inelastic term, or the cross section, or both. Finally, in Sec. 6 we use our methods to investigate the case of a polynomial amplitude. In particular, it is shown that a continuation away from both of the Crichton amplitudes is in general possible.

2. IMPLICIT FUNCTION THEOREM

Suppose that we know an elastically unitary scattering amplitude, $F_0(z)$, that is analytic in the small Martin ellipse, $\epsilon(z_0)$. Suppose that F_0 fits the given cross section, $\sigma_0(x)$, i.e.,

$$|F_0(x)|^2=\sigma_0(x), \tag{2.1}$$

for $-1\leqslant x\leqslant 1$. As in Ref. 11, one continues this equation into $\epsilon(z_0)$ by writing

$$\sigma_0(z)=F_0(z)F_0^*(z^*)=D_0^2(z)+A_0^2(z), \tag{2.2}$$

where D_0 and A_0 are the dispersive and absorptive parts of F_0, respectively. We wish to find sufficient conditions under which we may change $\sigma_0(z)$ to $\sigma_0(z)+\delta\sigma(z)$, and find a corresponding unitary amplitude $F_0(z)+\delta F(z)$.

In this general demonstration, we allow $D_0(z)$ to have zeros in $\epsilon(z_0)$. However, for the given $D_0(z)$, we can certainly find $\zeta_0'<z_0$, such that $|D_0(z)|$ does not vanish on $\partial\epsilon(\zeta_0')$. We shall define a set of real Hilbert spaces, $\mathcal{H}(\zeta)$, parametrized by the real number $\zeta>1$, by means of the inner product,

$$(f,g)_\zeta=\sum_{l=0}^{\infty}(2l+1)f_lg_l[Q_l(\zeta)]^{-2}, \tag{2.3}$$

on the set of functions, $f(z)$, that are real-analytic within $\epsilon(\zeta)$. Here

$$f_l=\tfrac{1}{2}\int_{-1}^{1}dz f(z)P_l(z), \tag{2.4}$$

and similarly for g. P_l and Q_l are the Legendre functions of the first and second kinds.

Note that an arbitrary function that is real-analytic in $\epsilon(\zeta)$ does not necessarily belong to $\mathcal{H}(\zeta)$, but it does belong to $\mathcal{H}(\eta)$, for any $\eta<\zeta$. Since $\sigma_0(z)$ is real-analytic in $\epsilon(z_0)$, it belongs to $\mathcal{H}(\zeta_0'')$, $\zeta_0''<z_0$, and for later convenience we choose ζ_0'' so that $\zeta_0'<\zeta_0''<z_0$. Because of the unitarity condition, we know that $A_0(z)$ is real-analytic in the *large* Martin ellipse, $\epsilon(2z_0^2-1)$. Hence it belongs to $\mathcal{H}(\zeta_1)$, where

$$\zeta_1=2\zeta_0^2-1, \tag{2.5}$$

where $\zeta_0<z_0$. For convenience we choose $\zeta_0<\zeta_0'$. Let us summarize the foregoing:

$$1<\zeta_0<\zeta_0'<\zeta_0''<z_0, \tag{2.6}$$

$$\sigma_0\in\mathcal{H}(\zeta_0'');\ D_0\in\mathcal{H}(\zeta_0'');\ A_0\in\mathcal{H}(\zeta_1). \tag{2.7}$$

We may define

$$n_0=\inf_{z\in\partial\epsilon(\zeta_0')}|D_0(z)|, \tag{2.8}$$

where it will be recalled that ζ_0' was chosen specifically to ensure that n_0 is strictly positive. We can arrange without difficulty that

$$\zeta_1 > \zeta_0''. \tag{2.9}$$

Let us define, for each pair

$$(A,\sigma) \in \mathcal{H}(\zeta_1) \times \mathcal{H}(\zeta_0''), \tag{2.10}$$

the following two sequences:

$$D_l = \tfrac{1}{2}\int_{-1}^{1} dx P_l(x)[\sigma(x) - A^2(x)]^{1/2}, \tag{2.11}$$

$$\overline{D}_l = \frac{1}{2\pi i} \oint_{\partial\epsilon(\zeta_0')} dz Q_l(z)[\sigma(z) - A^2(z)]^{1/2}, \tag{2.12}$$

$l = 0, 1, 2, \cdots$. In order to give a precise meaning to the integral (2.12), we must explain how to treat possible zeros of $\sigma(z) - A^2(z)$ within or upon the contour of integration, $\partial\epsilon(\zeta_0')$. Odd-order zeros will give rise to square-root branch points of the integrand of (2.12). In the case that the number of odd-order zeros in $\epsilon(\zeta_0')$ is even, suitable cuts may be drawn within the ellipse, and the integrand is then continuous around $\partial\epsilon(\zeta_0')$. If the number of odd-order zeros is odd, at least one such zero must lie on the real axis. In this case, we draw a cut from the rightmost real odd-order zero towards $+\infty$ along the real axis. Suitable cuts may be drawn within the ellipse between the other odd-order zeros. In this case the contour is open, but the integral is still well-defined, and moreover $\overline{D}_l$ is always purely real or purely imaginary. Hence $\overline{D}_l^2$ is well-defined and real.

The necessary and sufficient condition for the P_l-transform (2.11) and the Q_l-transform (2.12) to be equal is that there should be *no* odd-order zeros in $\epsilon(\zeta_0')$, so that the surd in (2.12) is analytic in $\epsilon(\zeta_0')$. We know from (2.7) that $D_0(z)$ is analytic in $\epsilon(\zeta_0')$, and from (2.2) that

$$D_0(z) = [\sigma_0(z) - A_0^2(z)]^{1/2}; \tag{2.13}$$

hence certainly

$$D_{0l} = \overline{D}_{0l}. \tag{2.14}$$

Let us define the nonlinear operator

$$M(A,\sigma; z) = \sum_{l=0}^{\infty} (2l+1)P_l(z)(A_l^2 + \overline{D}_l^2), \tag{2.15}$$

where $\overline{D}_l$ is given by (2.12), and where

$$A_l = \tfrac{1}{2}\int_{-1}^{1} dx P_l(x) A(x). \tag{2.16}$$

We will also use the operator

$$S(A,\sigma; z) = A(z) - M(A,\sigma; z). \tag{2.17}$$

Because $F_0(z)$ is an elastically unitary amplitude, and because of the equivalence (2.14), we know that

$$S(A_0, \sigma_0) = 0, \tag{2.18}$$

where we have suppressed the variable z. In this section, we shall use the Hildebrandt—Graves theorem to show that

$$S(A,\sigma) = 0 \tag{2.19}$$

defines $A(\sigma)$, as an implicit function of σ, for σ in some neighborhood of σ_0, such that

$$A(\sigma_0) = A_0. \tag{2.20}$$

This is not in itself enough to show that there exist unitary amplitudes, $F(\sigma)$, for σ in some neighborhood of σ_0, since in general

$$\overline{D}_l(\sigma) \neq D_l(\sigma). \tag{2.21}$$

We shall postpone until the next section the proof that there is an infinite-dimensional subset of the neighborhood of σ_0, in which indeed $\overline{D}_l$ and D_l are the same, so that unitarity is satisfied. For the present, we consider only Eq. (2.19), so we treat $\overline{D}_l$ only, and temporarily forget about D_l.

The form of the Hildebrandt—Graves theorem that we shall use, adapted to our particular case, is as follows (Ref. 33):

Let S be an operator taking pairs $(A,\sigma) \in \mathcal{H}(\zeta_1) \times \mathcal{H}(\zeta_0'')$ into $\mathcal{H}(\zeta_1)$. Suppose that $S(A_0,\sigma_0) = 0$, and that S is continuous with respect to (A,σ), in some neighborhood of (A_0,σ_0). Suppose also that $S_A(A,\sigma)$, the partial Fréchet derivative of S with respect to A, exists as a bounded linear operator on $\mathcal{H}(\zeta_1)$, and that it is continuous with respect to (A,σ) (in the operator topology) in the above neighborhood of (A_0,σ_0). Lastly, suppose that $S_A(A_0,\sigma_0)$ has an inverse, as a bounded linear operator on $\mathcal{H}(\zeta_1)$. Then $S(A,\sigma) = 0$ has a unique, continuous solution, $A(\sigma)$, for σ in some neighborhood of σ_0, with $A(\sigma_0) = A_0$.

This theorem is usually stated for Banach spaces rather than for Hilbert spaces. We understand that $\mathcal{H}(\zeta)$ is to be regarded as a Banach space, by means of the usual norm $(f,f)_\zeta^{1/2}$. We could have worked with the Banach space of Ref. 11, instead of $\mathcal{H}$, but we shall in fact find it very convenient to have a Hilbert space at our disposal, when we come to consider the singular case at the end of this section.

We need to check all the conditions of this theorem. Let us first prove that

$$(A,\sigma) \in \mathcal{H}(\zeta_1) \times \mathcal{H}(\zeta_0'') \Rightarrow M(A,\sigma) \in \mathcal{H}(\zeta_1). \tag{2.22}$$

Now $\sigma(z) - A^2(z)$ is analytic in $\epsilon(\zeta_0'')$, and hence

$$N^2 = \sup_{z \in \partial\epsilon(\zeta_0')} |\sigma(z) - A^2(z)| \tag{2.23}$$

is finite. We can deduce from (2.12) (see Ref. 11) that

$$|\overline{D}_l| \leq \frac{LN}{2\pi} Q_l(\zeta_0') \tag{2.24}$$

where L is the circumference of $\epsilon(\zeta_0')$. Also the fact that $\|A\|_{\zeta_1}$ exists implies that

$$|A_l| \leq \|A\|_{\zeta_1} Q_l(\zeta_1) < \|A\|_{\zeta_1} Q_l(\zeta_0'). \tag{2.25}$$

Hence

$$A_l^2 + \overline{D}_l^2 \leq \kappa[Q_l(\zeta_0')]^2 \leq \kappa\Omega(\zeta_1')Q_l(\zeta_1'), \tag{2.26}$$

where

$$\kappa = \frac{L^2N^2}{4\pi^2} + \|A\|_{\zeta_1}^2, \tag{2.27}$$

where

$$\zeta_1' = 2\zeta_0'^2 - 1 > \zeta_1, \tag{2.28}$$

and where

$$\Omega(\zeta_1') = [Q_0(\zeta_1')]^2 / Q_1(\zeta_1'), \tag{2.29}$$

as in Ref. 11. Hence

$$\|M(A,\sigma)\|_{\zeta_1}^2 = \sum_{l=0}^{\infty} (2l+1)[A_l^2 + \overline{D}_l^2]^2 [Q_l(\zeta_1)]^{-2}$$

$$\leq [\kappa\Omega(\zeta_1')]^2 \sum_{l=0}^{\infty} (2l+1)[Q_l(\zeta_1')/Q_l(\zeta_1)]^2. \tag{2.30}$$

This norm exists, since the series on the right converges exponentially, thanks to the inequality (2.28). This concludes the proof of (2.22).

Since $F_0(z)$ is analytic in $\epsilon(z_0)$, the zeros of $\sigma_0(z) - A_0^2(z)$ in $\epsilon(\zeta_0')$ are certainly of even order, and there can only be a finite number of them. If we change σ_0 to $\sigma_0 + \delta\sigma$, and wish to induce a change A_0 to $A_0 + \delta A$, consistent with Eq. (2.19), the zeros will in general move, and may split up into zeros of lower, and possibly odd order. Thus the surd $[\sigma(z) - A^2(z)]^{1/2}$ in (2.12) may no longer be analytic in $\epsilon(\zeta_0')$. This does not affect the applicability of the Hildebrandt–Graves theorem, provided we ensure that some neighborhood of $\partial\epsilon(\zeta_0')$ remains zero-free and is not intersected by square-root branch cuts. We will show that there is a neighborhood of (A_0, σ_0) in $\mathcal{H}(\zeta_1) \times \mathcal{H}(\zeta_0'')$, say Ξ, for which $\sigma(z) - A^2(z)$ does not vanish on $\partial\epsilon(\zeta_0')$. In fact, we shall take any n, such that $0 < n < n_0$, and show that a neighborhood Ξ exists, such that

$$n^2 \leq \inf_{\substack{z\in\partial\epsilon(\zeta_0')\\ (A,\sigma)\in\Xi}} |\sigma(z) - A^2(z)|. \tag{2.31}$$

Since $\sigma(z) - \sigma_0(z) \in \mathcal{H}(\zeta_0'')$, we see that for $z \in \epsilon(\zeta_0')$,

$$|\sigma(z) - \sigma_0(z)|^2 = |\sum_l (2l+1)P_l(z)[\sigma_l - \sigma_{0l}]|^2$$

$$\leq \sum_l (2l+1)\frac{|\sigma_l - \sigma_{0l}|^2}{[Q_l(\zeta_0'')]^2} \times \sum_l (2l+1)[P_l(\zeta_0')Q_l(\zeta_0'')]^2 \tag{2.32}$$

by the Schwarz inequality. The first sum here is $\|\sigma - \sigma_0\|^2_{\zeta_0''}$, and the second sum converges exponentially. Hence we have shown the existence of a constant, κ_1, such that

$$\sup_{z\in\epsilon(\zeta_0')} |\sigma(z) - \sigma_0(z)| \leq \kappa_1 \|\sigma - \sigma_0\|_{\zeta_0''}. \tag{2.33}$$

Similarly, we may show that there is a constant, κ_2, such that

$$\sup_{z\in\epsilon(\zeta_0')} |A(z) - A_0(z)| \leq \kappa_2 \|A - A_0\|_{\zeta_1}, \tag{2.34}$$

so that

$$\sup_{z\in\epsilon(\zeta_0')} |A^2(z) - A_0^2(z)|$$

$$\leq \sup_{z\in\epsilon(\zeta_0')} \{[|A(z) - A_0(z)| + 2|A_0(z)|]|A(z) - A_0(z)|\}$$

$$\leq \kappa_2^2[\|A - A_0\|_{\zeta_1} + 2\|A_0\|_{\zeta_1}]\|A - A_0\|_{\zeta_1}. \tag{2.35}$$

Hence

$$\sup_{z\in\epsilon(\zeta_0')} |[\sigma(z) - A^2(z)] - [\sigma_0(z) - A_0^2(z)]| \tag{2.36}$$

is bounded by the sum of the right-hand sides of (2.33) and (2.35). By making $\|\sigma - \sigma_0\|_{\zeta_0''}$ and $\|A - A_0\|_{\zeta_1}$ small enough, we may make (2.36) smaller than $n_0^2 - n^2$, and in view of (2.8) this suffices to demonstrate (2.31).

There is therefore a neighborhood, Ξ, of (A_0, σ_0), such that $\sigma(z) - A^2(z)$ remains zero-free for $z \in \partial\epsilon(\zeta_0')$. We may therefore assert, by the theorem of Rouché (Ref. 34), that the number of zeros of $\sigma(z) - A^2(z)$ inside $\partial\epsilon(\zeta_0')$ is the same as that of $\sigma_0(z) - A_0^2(z)$ (the zeros being counted according to their multiplicity). Any splitting of even-order zeros into odd-order zeros may be accommodated by cuts that do not intersect $\partial\epsilon(\zeta_0')$.

Let us now consider the Fréchet derivative

$$S_A(A, \sigma) = 1 - M_A(A, \sigma), \tag{2.37}$$

where

$$M_A(A, \sigma)\delta A = \sum_{l=0}^{\infty} (2l+1)P_l(z)\phi_l, \tag{2.38}$$

with

$$\phi_l = \frac{1}{i\pi}\oint_{\partial\epsilon(\zeta_0')} dt Q_l(t)\left(A_l - \frac{\overline{D}_l A(t)}{[\sigma(t) - A^2(t)]^{1/2}}\right)\delta A(t) \tag{2.39}$$

for any $\delta A \in \mathcal{H}(\zeta_1)$. We will now show that $M_A(A, \sigma)$ is a bounded linear operator on $\mathcal{H}(\zeta_1)$, if (A, σ) belongs to the neighborhood Ξ of (A_0, σ_0). In view of (2.24), (2.25) (2.31), and

$$\sup_{t\in\epsilon(\zeta_0')} |\delta A(t)| \leq \kappa_2 \|\delta A\|_{\zeta_1}, \tag{2.40}$$

which one can prove as in Eq. (2.34), we see that there is a constant, say κ_3, such that

$$|\phi_l| \leq \kappa_3[Q_l(\zeta_0')]^2 \|\delta A\|_{\zeta_1} \leq \kappa_3\Omega(\zeta_1')Q_l(\zeta_1')\|\delta A\|_{\zeta_1}, \tag{2.41}$$

ζ_1' being defined in (2.28). Hence [much as in Eq. (2.30)]

$$\|M_A(A, \sigma)\delta A\|^2_{\zeta_1} = \sum (2l+1)|\phi_l|^2[Q_l(\zeta_1)]^{-2}$$

$$\leq \{\kappa_3\Omega(\zeta_1')\|\delta A\|_{\zeta_1}\}^2 \sum_l (2l+1)[Q_l(\zeta_1')/Q_l(\zeta_1)]^2. \tag{2.42}$$

The series converges, and so we have proved that $M_A(A, \sigma)$ is a bounded linear operator. By similar, but somewhat longer calculations, it may be shown that $M_A(A, \sigma)$, and $S(A, \sigma)$ itself, are continuous with respect to (A, σ), for $(A, \sigma) \in \Xi$.

We have now verified all the conditions for the applicability of the Hildebrandt–Graves theorem, except the existence of an inverse linear operator $S_A^{-1}(A_0, \sigma_0)$. We show first that $M_A(A_0, \sigma_0)$ is compact, which we do by a method due to Johnson (Ref. 8). Let us define a ball of radius r in $\mathcal{H}(\zeta_1)$:

$$T_r = \{\alpha(z) : \|\alpha\|_{\zeta_1} \leq r, r > 0\}. \tag{2.43}$$

We prove that, for every $\epsilon > 0$, and any r, there exists a finite ϵ-net for the set $M_A(A_0, \sigma_0)T_r$, which means that it is totally bounded. For any $\delta A \in T_r$, we see from (2.41) that there is a constant, say κ_4, such that

$$|\phi_l| \leq \kappa_4 Q_l(\zeta_1'). \tag{2.44}$$

Hence, given any $\epsilon > 0$, we may find an L such that

$$\sum_{l=L}^{\infty} (2l+1)|\phi_l|^2[Q_l(\zeta_1)]^{-2} < \epsilon. \tag{2.45}$$

The L-tuples $(\phi_0, \phi_1, \ldots, \phi_{L-1})$, corresponding to all $\delta A \in T_r$, constitute a bounded set in the locally compact space $\mathbb{R}^L$, and so the set can be covered by a finite ϵ-net, which clearly also serves as a finite net for $M_A(A_0, \sigma_0)T_r$, in view of (2.45). Hence $M_A(A_0, \sigma_0)$ is a compact linear operator on $\mathcal{H}(\zeta_1)$, and we may apply the Riesz–Schauder theory (Ref. 35). In particular, the spectrum is a point set, and if unity does not belong to it, $S_A(A_0, \sigma_0)$ has a bounded inverse, and the Hildebrandt–Graves theorem applies. If unity *is* an eigenvalue of $M_A(A_0, \sigma_0)$, nevertheless the corresponding eigenspace is only finite-dimensional, by Riesz—Schauder, and we will show that Eq. (2.19) can still be used in general to define an implicit function, $A(\sigma)$.

We attack the singular case, when unity is an eigenvalue of $M_A(A_0, \sigma_0)$, by a modification of bifurcation theory (Ref. 36). We define

$$U(\delta A, \delta\sigma) = S(A, \sigma) - S_A(A_0, \sigma_0)\delta A, \tag{2.46}$$

where

$$\delta A = A - A_0, \tag{2.47a}$$

$$\delta\sigma = \sigma - \sigma_0. \tag{2.47b}$$

Again we start from (2.18), and we wish to demonstrate the existence of a solution, $A(\sigma)$, of Eq. (2.19), for $\sigma \neq \sigma_0$. At such a solution,

$$S_A(A_0, \sigma_0)\delta A = -U(\delta A, \delta\sigma), \tag{2.48}$$

but now we assume that unity is in the spectrum of $M_A(A_0, \sigma_0)$, so that $S_A(A_0, \sigma_0)$ has no inverse. We will show nevertheless that there does exist a so-called pseudoinverse. We know that $M_A(A_0, \sigma_0)$ is compact, and this allows us to assert two things:

(a) the nullspace of $S_A(A_0, \sigma_0)$, say N, is a linear subspace of $\mathcal{H}(\zeta_1)$ of finite dimension, say n;

(b) the range of $S_A(A_0, \sigma_0)$, say R, is strongly closed.

These results follow from the Riesz—Schauder theory.

We define the quotient space

$$\mathcal{H}_N(\zeta_1) = \mathcal{H}(\zeta_1)/N \tag{2.49}$$

in the standard way. This is a Banach space, normed by

$$\| \delta A_N \|_N = \inf_{\alpha \in N} \| \delta A + \alpha \|_{\zeta_1} \tag{2.50}$$

where δA_N is the equivalence class, modulo N, that contains δA (i.e., $\delta A \in \delta A_N$ and $x \in \delta A_N$, $y \in \delta A_N \Rightarrow x - y \in N$). There exists a continuous, linear one-to-one mapping, from $\mathcal{H}_N(\zeta_1)$ to R, say S_N, such that

$$S_N \delta A_N = S_A(A_0, \sigma_0)\delta A. \tag{2.51}$$

Then the inverse mapping theorem tells us that S_N has a continuous inverse, S_N^{-1}, as a linear mapping from R to $\mathcal{H}_N(\zeta_1)$. Let Π_N and Π_R be the orthogonal projection operators from $\mathcal{H}(\zeta_1)$ onto N and R, respectively. Then the pseudoinverse of $S_A(A_0, \sigma_0)$ is defined, as a bounded linear mapping from $\mathcal{H}(\zeta_1)$ to $N^\perp$, the subspace of $\mathcal{H}(\zeta_1)$ orthogonal to N, by

$$\tilde{S}_A^{-1} = (I - \Pi_N)_N S_N^{-1} \Pi_R, \tag{2.52}$$

where $(I - \Pi_N)_N$ is the linear mapping from $\mathcal{H}_N(\zeta_1)$ to $N^\perp$ defined by

$$(I - \Pi_N)_N \delta A_N = (I - \Pi_N)\delta A. \tag{2.53}$$

Clearly $(I - \Pi_N)_N$ is an isometry.

The pseudoinverse, $\tilde{S}_A^{-1}$, will now serve to transform Eq. (2.48), but only on condition that $U(\delta A, \delta\sigma)$ belongs to R. Consider in fact the equation

$$\delta A = -\tilde{S}_A^{-1} U(\delta A, \delta\sigma) + u, \tag{2.54}$$

where $u \in N$. We may write

$$u = \sum_{m=1}^{n} \lambda_m u_m \tag{2.55}$$

where $\{u_m\}$ is a real basis for N, and $\{\lambda_m\}$ is a set of real numbers. We shall show presently that, if $\|\delta\sigma\|_{\zeta_0''}$ and $\|u\|_{\zeta_1}$ are small enough, Eq. (2.54) defines a contraction mapping, so that then a locally unique solution, say $\delta A(\delta\sigma, u)$, exists. Now from (2.52) we have that

$$S_A(A_0, \sigma_0)\tilde{S}_A^{-1} = \Pi_R \tag{2.56}$$

so that (2.54) implies

$$S_A(A_0, \sigma_0)\delta A = -\Pi_R U(\delta A, \delta\sigma), \tag{2.57}$$

and this reduces to (2.48) only if

$$(1 - \Pi_R) U(\delta A(\delta\sigma, u), \delta\sigma) = 0, \tag{2.58}$$

as expected.

The system (2.58) is in fact of dimension n, since $(1 - \Pi_R)$ is the projection operator onto the nullspace of the adjoint operator $S_A^\dagger(A_0, \sigma_0)$. Since u depends on the n real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ [Eq. (2.55)], we may regard (2.58) as a system of n nonlinear algebraic equations for the n variables λ_m, with $\delta\sigma$ as an infinite-dimensional parameter. This system is called the bifurcation equation, and in general there will be more than one solution for the λ_m. Clearly only real solutions are of interest: complex ones are simply to be ignored. For each real solution for the λ_m, there corresponds a solution $u(\delta\sigma)$ of the bifurcation equation (2.58), and for this solution, the function

$$\delta A(\delta\sigma, u(\delta\sigma)) \tag{2.59}$$

solves not only (2.54) [with $u = u(\delta\sigma)$], but also the original equation (2.48).

It remains to supply the contraction mapping proof. Define the nonlinear mapping F on $\mathcal{H}(\zeta_1)$:

$$F(\delta A) = -\tilde{S}_A^{-1} U(\delta A, \delta\sigma) + u. \tag{2.60}$$

Now according to (2.46) and (2.18),

$$U(\delta A, \delta\sigma) = S(A_0 + \delta A, \sigma_0 + \delta\sigma) - S(A_0, \sigma_0) - S_A(A_0, \sigma_0)\delta A \tag{2.61}$$

and hence we infer that

$$\begin{aligned} \| U(\delta A, \delta\sigma) \|_{\zeta_1} \leq & \| S(A_0 + \delta A, \sigma_0 + \delta\sigma) - S(A_0 + \delta A, \sigma_0) \|_{\zeta_1} \\ & + \| S(A_0 + \delta A, \sigma_0) - S(A_0, \sigma_0) \\ & - S_A(A_0, \sigma_0)\delta A \|_{\zeta_1}. \end{aligned} \tag{2.62}$$

To bound the first term on the right-hand side of (2.62), we use the Banach space version of the mean-value theorem:

$$\| S(A, \sigma_0 + \delta\sigma) - S(A, \sigma_0) \|_{\zeta_1} \leq \sup_{0 \leq x \leq 1} \| S_\sigma(A, \sigma_0 + x\delta\sigma \| \circ \| \delta\sigma \|_{\zeta_0''}, \tag{2.63}$$

where $S_\sigma(A, \sigma)$ is the partial Fréchet derivative of $S(A, \sigma)$ with respect to σ. This is defined by

$$S_\sigma(A, \sigma)\delta\sigma = \sum_{l=0}^{\infty} (2l+1) P_l(z) \psi_l, \tag{2.64a}$$

in which

$$\psi_l = \frac{\overline{D}_l}{2i\pi} \oint_{\partial\epsilon(\zeta_0')} dt Q_l(t) \frac{\delta\sigma(t)}{[\sigma(t) - A^2(t)]^{1/2}}, \tag{2.64b}$$

and where $\delta\sigma \in \mathcal{H}(\zeta_0'')$. It is easy to show that $S_\sigma(A,\sigma)$ is a bounded, and in fact a compact linear operator from $\mathcal{H}(\zeta_0'')$ to $\mathcal{H}(\zeta_1)$, if $(A,\sigma)\in\Xi$. The method is precisely similar to the demonstration that $M_A(A,\sigma)$ is compact, and we leave the details to the reader [see Eq. (2.38) *et seq.*]. Hence there is a constant, C_1, such that the right-hand side of (2.63) is bounded by $C_1\|\delta\sigma\|_{\zeta_0''}$, so long as $(A,\sigma)\in\Xi$.

We turn now to the second term on the right-hand side of (2.62). Here we shall use the Banach space version of the second mean-value theorem, viz.,

$$\|S(A,\sigma_0) - S(A_0,\sigma_0) - S_A(A_0,\sigma_0)\delta A\| \le \tfrac{1}{2}\sup_{0\le x\le 1}\|S_{AA}(A_0 + x\delta A,\sigma_0)\|\cdot\|\delta A\|^2 \tag{2.65}$$

where all norms refer to $\mathcal{H}(\zeta_1)$. Here the second partial Fréchet derivative with respect to A, $S_{AA}(A,\sigma)$, may be shown to be a bounded bilinear operator from $\mathcal{H}(\zeta_1)\times\mathcal{H}(\zeta_1)$ to $\mathcal{H}(\zeta_1)$, if $(A,\sigma)\in\Xi$, by methods similar to those used to show that $M_A(A,\sigma)$ is bounded. Hence there exists a constant C_2, such that the right-hand side of (2.65) is bounded by $C_2\|\delta A\|^2_{\zeta_1}$ (see Ref. 13 for a discussion of the second Fréchet derivative in a mathematically similar problem).

From the definition (2.60), we may therefore write the inequality

$$\|F(\delta A)\|_{\zeta_1} \le \|\tilde{S}_A^{-1}\|\{C_1\|\delta\sigma\|_{\zeta_0''} + C_2\|\delta A\|^2_{\zeta_1}\} + \|u\|_{\zeta_1}. \tag{2.66}$$

We have already shown that the pseudoinverse, $\tilde{S}_A^{-1}$, is a bounded linear operator, so it suffices to take

$$\|\delta A\|_{\zeta_1} \le b, \tag{2.67a}$$

$$\|\delta\sigma\|_{\zeta_0''} \le b[3\|\tilde{S}_A^{-1}\|C_1]^{-1}, \tag{2.67b}$$

$$\|u\|_{\zeta_1} \le b/3, \tag{2.67c}$$

where b is a number that satisfies

$$b \le [3\|\tilde{S}_A^{-1}\|C_2]^{-1}, \tag{2.68}$$

and is small enough to ensure that (2.67a,b) implies $(A,\sigma)\in\Xi$. The above conditions are sufficient to give

$$\|F(\delta A)\|_{\zeta_1} \le b, \tag{2.69}$$

so F is an injective mapping of the ball (2.67a) into itself.

To show that F is contractive, and not merely injective on the ball (2.67a), we consider

$$F(\delta A_1) - F(\delta A_2) = -\tilde{S}_A^{-1}\{U(\delta A_1,\delta\sigma) - U(\delta A_2,\delta\sigma)\} \tag{2.70}$$

for any δA_1 and δA_2 in the ball (2.67a). So

$$\begin{aligned}\|F(\delta A_1) - F(\delta A_2)\|_{\zeta_1} &\le \|\tilde{S}_A^{-1}\|\{\|S(A_1,\sigma) - S(A_1,\sigma_0) - S(A_2,\sigma) + S(A_2,\sigma_0)\|_{\zeta_1} \\ &\quad + \|S(A_1,\sigma_0) - S(A_2,\sigma_0) - S_A(A_0,\sigma_0)(A_1 - A_2)\|_{\zeta_1}\}.\end{aligned} \tag{2.71}$$

The first term within the parentheses may be bounded by

$$\sup_{0\le x\le 1}\sup_{0\le y\le 1}\|S_{A\sigma}(A_1 + x(A_2 - A_1),\sigma_0 + y\delta\sigma)\|\cdot\|\delta\sigma\|_{\zeta_0''}\|A_1 - A_2\|_{\zeta_1} \tag{2.72}$$

where $S_{A\sigma}$ is the mixed second-order Fréchet derivative of $S(A,\sigma)$ with respect to A and σ. It may be shown to be bounded. The second term within the parentheses is bounded by

$$\tfrac{1}{2}\sup_{0\le x\le 1}\|S_{AA}(A_1 + x(A_2 - A_1),\sigma_0)\|\cdot\|A_1 - A_2\|^2_{\zeta_1}. \tag{2.73}$$

Hence there are constants, C_3 and C_4, such that

$$\|F(\delta A_1) - F(\delta A_2)\|_{\zeta_1} \le \kappa\|A_1 - A_2\|_{\zeta_1} \tag{2.74}$$

with

$$\kappa = C_3\|\delta\sigma\|_{\zeta_0''} + C_4 b. \tag{2.75}$$

Clearly it is possible to choose b so small that simultaneously (2.67b) and (2.68) are satisfied, and $\kappa < 1$. This is the condition for a contraction, and with it we have ended the proof.

3. ZEROS OF THE DISPERSIVE PART

In this section we consider how to ensure that

$$\overline{D}_l(\sigma) = D_l(\sigma), \tag{3.1}$$

so that elastic unitarity is satisfied by the implicit function $D(\sigma;z) + iA(\sigma;z)$, the existence of which we demonstrated in Sec. 2. We have shown that, if $\|\sigma - \sigma_0\|_{\zeta_0''}$ is small enough, the algebraic number of zeros of

$$R(A,\sigma;z) = \sigma(z) - A^2(z) \tag{3.2}$$

within $\epsilon(\zeta_0')$ is constant, where we understand that $A(z)$ in (3.2) is the implicit function defined by

$$S(A,\sigma;z) = 0. \tag{3.3}$$

Suppose that $R(A_0,\sigma_0;z)$ has N zeros within $\epsilon(\zeta_0')$, at the positions $z = p_{01}, p_{02}, \ldots, p_{0N}$, and that the orders of the zeros are respectively $2q_1, 2q_2, \ldots, 2q_N$ [the orders must be even, since $D_0(z) = R^{1/2}(A_0,\sigma_0;z)$ is analytic in $\epsilon(\zeta_0')$]. Then we know that

$$R^{(m)}(A_0,\sigma_0;p_{0n}) = 0 \tag{3.4}$$

where $m = 0,1,\ldots,2q_n - 1$, $n = 1,2,\ldots,N$, and where

$$R^{(m)}(A,\sigma;z) = \left(\frac{\partial}{\partial z}\right)^m R(A,\sigma;z) \tag{3.5}$$

(with the understanding of course that $R^{(m)}$ just reduces to R for $m = 0$). We shall show that σ can be so constrained that also

$$R^{(m)}(A,\sigma;p_n) = 0 \tag{3.6}$$

for the same values of m and n. In other words, the zeros of $R(A,\sigma;z)$ have moved from the old positions, p_{0n}, to new positions p_n, and are of the same (even) order as before. Thus $D(z)$ is still analytic and (3.1) is guaranteed.

To make this quite precise, we shall ensure that all the p_n, $n = 1,2,\ldots,N$, remain distinct, so that then the zero at p_n is precisely of order $2q_n$ [for if it were of even higher order, the algebraic number of zeros of $R(A,\sigma;z)$ within $\epsilon(\zeta_0')$ would be greater than $2(q_1 + q_2 + \cdots + q_N)$, which is impossible].

We simply need to apply the implicit-function theorem to the finite-dimensional system

$$R^{(m)}(A(\sigma),\sigma;p_n) = 0 \tag{3.7}$$

for $m = 0,1,\ldots,2q_n - 1$, $n = 1,2,\ldots,N$, where $A(\sigma)$ is the implicit function defined by (3.3). There are in fact $2(q_1+q_2+\cdots+q_N)$ complex equations, and so we cannot hope to use (3.7) to define the N p's as implicit functions of an unrestricted σ. We shall in fact write

$$\sigma(z)=\sigma_f(z)+\sigma_c(z), \qquad (3.8)$$

where we define the "constrained part" of the cross section by

$$\sigma_c(z)=\sum_{k=1}^{Q}(2l_k+1)\sigma_{l_k}P_{l_k}(z), \qquad (3.9a)$$

where $l_1, l_2, \ldots, l_Q$, is an arbitrary sequence of integers (we shall define Q presently), and where the "free part" of the cross section is

$$\sigma_f(z)=\sum_{l\neq l_k}(2l+1)\sigma_l P_l(z), \quad k=1,2,\ldots,Q \qquad (3.9b)$$

We may then write (3.7) as

$$R^{(m)}(A(\sigma_f+\sigma_c),\sigma_f+\sigma_c;p_n)=0. \qquad (3.10)$$

Although there are $4(q_1+\cdots+q_N)$ real equations here, only half this number are independent, since complex p's must come in complex conjugate pairs, and for real p's, the corresponding equation is manifestly real. Accordingly, it is sufficient to consider only zeros on the real axis, or in the upper half-plane [and inside $\epsilon(\zeta_0'')$], and to define

$$Q=2(q_1+\cdots+p_N)-N. \qquad (3.11)$$

Then (3.10) is a system of $Q+N$ real scalar equations for $Q+N$ real unknowns (namely the σ_{l_k} and the real and imaginary parts of the p's in the upper half-plane). The equations are to be solved for these unknowns, in terms of σ_f, which may be chosen freely.

The question of the existence of solutions of (3.10), for σ_f sufficiently close to σ_{0f}, can be answered by another application of the implicit function theorem.

The derivatives of $R^{(m)}(A(\sigma_f+\sigma_c),\sigma_f+\sigma_c;p_n)$, with respect to σ_c and p_n, are easy to calculate, and may be shown to be bounded, and continuous with respect to σ_f, σ_c, and p_n by methods following closely those of Sec. 2. The ordinary implicit function theorem is applicable if the derivative system has an inverse; if it does not, we can treat the bifurcation equation, as in Sec. 2.

The shifts in the positions of the zeros, $|p_n-p_{0n}|$, are proportional to $\|\sigma_f-\sigma_{0f}\|$, and so by making the latter quantity small enough, we can ensure that none of the zeros move by more than (say), one-third of the distance between the closest pair of zeros. In this way we can be sure that all zeros remain distinct, and that the orders, $2q_n$, do not change.

4. PRACTICAL IMPLEMENTATION OF THE METHOD

In practice, we set up a modified Newton-Kantorovich iteration for the numerical calculation of the new amplitudes that correspond to the changed cross-sections. We shall simply write down the equations and refer the reader to Ref. 14 for a detailed discussion of the method.

$$S_l(A^{(j)},\sigma_c^{(j)}+\sigma_{0f}+\delta\sigma_f)+\sum_{r=0}^{\infty}\frac{\partial S_l(A^{(0)},\sigma_c^{(0)}+\sigma_{0f}+\delta\sigma_f)}{\partial A_r}\delta A_r^{(j+1)}$$
$$+\sum_{k=1}^{Q}\frac{\partial S_l(A^{(0)},\sigma_c^{(0)}+\sigma_{0f}+\delta\sigma_f)}{\partial\sigma_{l_k}}\delta\sigma_{l_k}^{(j+1)}=0, \qquad (4.1a)$$

$$R^{(m)}(A^{(j)},\sigma_c^{(j)}+\sigma_{0f}+\delta\sigma_f;p_n^{(j)})$$
$$+\sum_{r=0}^{\infty}\frac{\partial R^{(m)}(A^{(0)},\sigma_c^{(0)}+\sigma_{0f}+\delta\sigma_f,p_n^{(0)})}{\partial A_r}\delta A_r^{(j+1)}$$
$$+\sum_{k=1}^{Q}(2l_k+1)P_{l_k}^{(m)}(p_n^{(0)})\delta\sigma_{l_k}^{(j+1)}+\delta\sigma_f^{(m+1)}(p_n^{(0)})\delta p_n^{(j+1)}$$
$$+\delta_{m,2q_n-1}R^{(m+1)}(A^{(0)},\sigma_c^{(0)}+\sigma_{0f};p_n^{(0)})\delta p_n^{(j+1)}=0. \qquad (4.1b)$$

Here S_l is the partial-wave projection of S, i.e.,

$$S_l(A,\sigma_c+\sigma_f)=A_l-A_l^2-\overline{D}_l^2(A,\sigma_c+\sigma_f). \qquad (4.2)$$

For numerical convenience, we work in Eq. (4.1) directly with the partial waves A_l and σ_l, rather than with the functions $A(z)$ and $\sigma(z)$. The partial-wave index l runs from 0 to ∞, although in practice this means 0 to some sufficiently large $L_{\max}$, at which point the r-series in (4.1) are also cut off. In (4.1b), m runs over $0,2,\ldots,2q_n-1$, and n over the subset of $1,2,\ldots,N$ that corresponds to the zeros in the upper half-plane, or on the real axis, and within the ellipse $\epsilon(\zeta_0'')$. We have used the notation

$$P_l^{(m)}(z)=\left(\frac{d}{dz}\right)^m P_l(z), \qquad (4.3)$$

$$\delta\sigma_f^{(m+1)}(z)=\left(\frac{d}{dz}\right)^{m+1}\delta\sigma_f(z)$$
$$=\sum_{l\neq l_k}(2l+1)\delta\sigma_{fl}P_l^{(m+1)}(z), \quad k=1,2,\ldots,Q, \qquad (4.4)$$

and we have taken account of the fact that

$$R^{(m)}(A^{(0)},\sigma_c^{(0)}+\sigma_{0f};p_n^{(0)})=0 \qquad (4.5)$$

for $m=0,1,\ldots,2q_n-1$, $n=1,2,\ldots,N$, in order to simplify the p_n-derivative term in (4.1b). Finally, j labels the iteration step, and we define

$$\begin{aligned}\delta A_r^{(j+1)}&=A_r^{(j+1)}-A_r^{(j)},\\ \delta\sigma_{l_k}^{(j+1)}&=\sigma_{l_k}^{(j+1)}-\sigma_{l_k}^{(j)},\\ \delta p_n^{(j+1)}&=p_n^{(j+1)}-p_n^{(j)}.\end{aligned} \qquad (4.6)$$

The Kantorovich theorem[33] guarantees the convergence of the iteration (4.1), if the corresponding inverses exist,[14] since it may be shown that S and $R^{(m)}$ are twice Fréchet differentiable with respect to A and σ_c. In the event that (4.1) cannot be inverted to give the quantities (4.6), one may approximate the bifurcation equation by extending the Newton expansion to second order. We refer to Refs. 11 and 36 for further details.

5. EXTENSION TO INELASTIC UNITARITY

In Secs. 3 and 4 we discussed at length the construction of new amplitudes corresponding to changed cross sections. The continuum ambiguity resulting from changes in the inelastic contributions to the amplitude has been treated in Refs. 11 and 14. In this section we extend these methods is such a way as to allow us to change the cross sections and the inelasticities simultaneously.

At energies above the first inelastic threshold we write the unitarity condition

$$A_l = A_l^2 + D_l^2 + I_l, \tag{5.1}$$

where the inelasticities I_l have to satisfy the inequality

$$0 \leq I_l \leq \tfrac{1}{4}. \tag{5.2}$$

Following the technique of Sec. 2, we define an operator S, which will now also depend upon the I_l's,

$$S(A, \sigma, I; z) = A(z) - M(A, \sigma; z) - I(z) \tag{5.3}$$

where

$$I(z) = \sum_l (2l+1) I_l P_l(z). \tag{5.4}$$

Because of (5.1) and the equivalence (2.14) we know that

$$S(A_0, \sigma_0, I_0; z) = 0, \tag{5.5}$$

where $I_0(z)$ is the inelastic contribution to the amplitude $F_0(z)$. We now change the differential cross section $\sigma_0(z)$ and the inelastic part $I_0(z)$ by small amounts $\delta\sigma(z)$ and $\delta I(z)$, and we want to construct a new absorptive part $A(z)$ such that

$$S(A, \sigma_0 + \delta\sigma, I_0 + \delta I) = 0 \tag{5.6}$$

where we have suppressed the variable z. The proof that such an $A(z)$ exists is essentially the same as that given in Sec. 2. Again we have the problem of the zeros of the dispersive part. To prevent the zeros from parasitizing, we have to ensure, as in Sec. 3, that (3.6) is satisfied. However, we now have the choice of constraining either part of $\sigma(z)$, or part of $I(z)$, or both. We therefore write the following system of equations

$$S(A, \sigma_c + \sigma_f, I_c + I_f) = 0, \tag{5.7}$$

$$R^{(m)}(A(\sigma_f + \sigma_c, I_f + I_c), \sigma_f + \sigma_c; p_n) = 0 \tag{5.8}$$

and apply a modified Newton—Kantorovich iteration in order to solve them. The final equations are then similar to Eqs. (4.1) except for a term

$$-\sum_{n=1}^{R} \delta_{l,n} \delta I_{l_n}^{(j+1)} \tag{5.9}$$

in Eq. (4.1a) where R is the number of constrained I_l's and the index k in (4.1a) and (4.1b) runs from 1 to $Q-R$.

6. NEIGHBORHOODS OF THE CRICHTON AMBIGUITY

In this section, we shall first discuss in more detail the nature of the singularities of $\partial S/\partial A$. Then we shall illustrate the foregoing ideas by considering the case of a polynomial amplitude, in particular, certain aspects of the Crichton ambiguity will be elucidated.

As in the previous section, we set

$$S_l = A_l - A_l^2 - D_l^2 - I_l \tag{6.1}$$

where I_l is zero if the energy is below the first inelastic threshold, and is otherwise bounded between 0 and $\frac{1}{4}$. Then we may write the partial Fréchet derivative

$$\frac{\partial S_l}{\partial A_m} = [1 - 2A_l]\delta_{lm} - 2D_l \frac{\partial D_l}{\partial A_m} \tag{6.2}$$

with

$$\frac{\partial D_l}{\partial A_m} = -\frac{2m+1}{2\pi i} \oint_{\partial\epsilon(\zeta_0')} dz Q_l(z) P_m(z) \frac{A(z)}{D(z)}. \tag{6.3}$$

We know that $\partial S_l/\partial A_m - \delta_{lm}$ is a compact linear operator on the Hilbert space $\mathcal{H}(\zeta_1)$ of Sec. 2, and therefore $\partial S_l/\partial A_m$ will fail to have an inverse precisely when there exists a nonzero sequence $\{\alpha_l\}$, belonging to $\mathcal{H}(\zeta_1)$, for which

$$\sum_{m=0}^{\infty} \frac{\partial S_l}{\partial A_m} \alpha_m = 0. \tag{6.4}$$

If we consider only first order changes in A and σ, we may write for (2.19)

$$\frac{\partial S}{\partial A} \delta A + \frac{\partial S}{\partial \sigma} \delta\sigma = 0 \tag{6.5}$$

where we have not made the distinction between $\delta\sigma_c$ and $\delta\sigma_f$ explicit, and where the partial wave subscripts and summations are also implicit. If

$$\frac{\partial S}{\partial A} \alpha = 0 \tag{6.6}$$

which is (6.4) written implicitly, and if the normalized null function α is unique, then we impose the following linear constraint on $\delta\sigma$:

$$\left(\alpha, \frac{\partial S}{\partial \sigma} \delta\sigma\right)_{\zeta_1} = 0 \tag{6.7}$$

and then solve (6.5) as

$$\delta A = -\tilde{S}_A^{-1} \frac{\partial S}{\partial \sigma} \delta\sigma + \lambda\alpha \tag{6.8}$$

where $\tilde{S}_A^{-1}$ is the pseudoinverse that was introduced in Sec. 2, and (6.8) is the first-order version of (2.54). The number λ has to be determined from the bifurcation equation (2.58), which may be approximated by extending (6.5) to second order in δA, and by contracting this against α. It is easy to see that, as $\delta\sigma \to 0$, λ is generally of a lower order (apart from exceptional cases, λ is of order $\delta\sigma^{1/2}$). Hence, to order λ only, $\delta A = \lambda\alpha$, and since

$$\delta D = (\delta\sigma - 2A\delta A)/2D, \tag{6.9}$$

it follows that

$$\delta D = -\lambda A \alpha / D, \tag{6.10}$$

to order λ. If we have a bifurcation point of the complete system (3.3) and (3.7), then δD must be analytic within the ellipse of integration $\partial\epsilon(\zeta_0')$. Therefore any zeros of D within $\partial\epsilon(\zeta_0')$ which are not cancelled by zeros of A must be cancelled by zeros of α.

One interesting property of α immediately follows from the analyticity of δD within $\partial\epsilon(\zeta_0')$. Using (6.2) and (6.3), we write for (6.4)

$$(1 - 2A_l)\alpha_l + \frac{D_l}{\pi i} \oint_{\partial\epsilon(\zeta_0')} dz Q_l(z) \frac{A(z)\alpha(z)}{D(z)} = 0 \tag{6.11}$$

where

$$\alpha(z) = \sum_{m=0}^{\infty} (2m+1)\alpha_m P_m(z). \tag{6.12}$$

Because of (6.10) and the analyticity of δD, we can distort the contour $\partial\epsilon(\zeta_0')$ and squeeze it around the cut $(-1, +1)$ to obtain

$$(1-2A_l)\alpha_l + D_l\int_{-1}^{+1} dx P_l(x)\frac{A(x)\alpha(x)}{D(x)} = 0.$$

If we now multiply this by $(2l+1)$ and sum over l, we find

$$\alpha(1) = 0 \tag{6.13}$$

so that α vanishes in the forward direction.

Let us conclude this general discussion by listing sufficient conditions such that $F = D + iA$ is a bifurcation point of the complete system.

(a) there is a nontrivial solution α of (6.4);

(b) $A(z)\alpha(z)/D(z)$ is analytic within $\epsilon(\zeta_0')$;

(c) $\partial^2 S(A,\sigma)/\partial A^2 \cdot \alpha \cdot \alpha \neq 0$;

(d) F satisfies unitarity and $|F|^2 = \sigma$.

Now we shall consider the case that both $A(z)$ and $D(z)$ are polynomials of degree L, and that all zeros of $D(z)$ are simple and lie within the ellipse $\partial\epsilon(\zeta_0')$. We may write

$$\frac{A(z)}{D(z)} = \gamma_0 + \sum_{s=1}^{L}\frac{\gamma_s}{z-\beta_s} \tag{6.14}$$

where the β's are the zeros of $D(z)$. We may identify γ_0 by considering the limit $z\to\infty$ in (6.14):

$$\gamma_0 = A_L/D_L. \tag{6.15}$$

We now evaluate the integral (6.3) as follows:

$$\frac{\partial D_l}{\partial A_m} = -\frac{A_L}{D_L}\delta_{lm} - \theta(m-l-1)(2m+1) \times \sum_{s=1}^{L}\gamma_s[Q_l(\beta_s)P_m(\beta_s) - P_l(\beta_s)Q_m(\beta_s)] \tag{6.16}$$

and express the condition (6.4) in the form

$$\left[1 - 2A_l + 2\frac{A_L}{D_L}D_l\right]\alpha_l = -2D_l\sum_{m=l+1}^{\infty}(2m+1)\alpha_m \times \sum_{s=1}^{L}\gamma_s[Q_l(\beta_s)P_m(\beta_s) - P_l(\beta_s)Q_m(\beta_s)]. \tag{6.17}$$

For $l > L$ we have $D_l = 0 = A_l$, and therefore $\alpha_l = 0$. Hence we deduce from (6.17) in the case $l = L$ that $\alpha_L = 0$. The sum (6.4) is thus automatically truncated at $l = L-1$ or lower. For $l = L-1$ one finds

$$\alpha_{L-1} = 0 \quad \text{or} \quad 1 - 2A_{L-1} + 2A_L D_{L-1}/D_L = 0. \tag{6.18}$$

In the first case one then has

$$\alpha_{L-2} = 0 \quad \text{or} \quad 1 - 2A_{L-2} + 2A_L D_{L-2}/D_L = 0 \tag{6.19}$$

and so on. If

$$1 - 2A_l + 2A_L D_l/D_L \neq 0 \tag{6.20}$$

for $l = 0, 1, 2, \ldots, L-1$, then $\alpha_l = 0$, all l, which means that there is no nontrivial null sequence, and hence that $\partial S_l/\partial A_m$ is nonsingular and has an inverse. If, on the other hand,

$$1 - 2A_l + 2A_L D_l/D_L = 0 \tag{6.21}$$

with $l = n$, for one and only one integer n between 0 and $L-1$, then there is just one independent null function, which one may obtain from (6.17) by setting $\alpha_n = 1$, and then by solving successively for $\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_0$. In the case that (6.21) is satisfied for more than one value of l, there may in general be more than one independent null function; but we shall not examine this case further.

For purely elastic amplitudes, the condition (6.21) for a singularity of $\partial S/\partial A$ can be reduced to:

$$2\delta_l = [(2N+1)/2]\pi + \delta_L \tag{6.22}$$

where the δ_l's are the real phase-shifts and N is an integer. There are many polynomial amplitudes that have one or more partial waves satisfying (6.22). However, the subsidiary condition that $\alpha A/D$ be analytic is met only in a few cases. Explicit examples of these are given by the endpoints of the SPD[26] and SPDF[28] Crichton-like ambiguities. Crichton-like ambiguities exist whenever there are two polynomial amplitudes F and F', with the same fixed inelasticities, each of degree L, that have the same modulus. It has been shown in the $L = 2$ and $L = 3$ cases, that the amplitudes F and F' are on closed curves in the space spanned by $A_0, A_1, \ldots, A_L$. These curves can be parametrized by the change in the differential cross section σ. The endpoints of the ranges of σ for these curves correspond precisely to bifurcation points of the complete system. In the SPD Crichton case there are in fact two singular points, in addition to the bifurcation points (which occur at $\delta_2 = 12.5°$ and $\delta_2 = 24.2°$). They correspond to the satisfaction of Eq. (6.22) for the S wave: for $\delta_2 = 13.5°$, $\partial S/\partial A$ is singular at F, and for $\delta_2 = 23.0°$, it is singular at F'.

Given F and F' with the same σ it is generally possible to construct two new amplitudes $F + \delta F$ and $F' + \delta F'$ corresponding to the same slightly changed cross section $\sigma + \delta\sigma$ by a simple modification of the method of Sec. 4. In particular, it is possible to follow the Crichton curves mentioned above, and we shall outline the method for the case $L = 2$. In this case D and D' each have two real simple zeros, and therefore two of the $\delta\sigma_l$ must be constrained in order to prevent the zeros from parasitizing. These constraints are different for D and D', and therefore in general we would expect $\delta\sigma_c \neq \delta\sigma_c'$. Nevertheless, we now show that it is possible to ensure that $\delta\sigma_c = \delta\sigma_c'$, and thus to ensure that $|F + \delta F| = |F' + \delta F'|$. Indeed the cross section $\sigma(z)$ is a fourth-order polynomial, so there are five Legendre coefficients, $\sigma_0, \sigma_1, \ldots, \sigma_4$; so that it suffices to take four of the σ_l, instead of two, as members of the constrained set σ_c. If we exclude the singular points of $\partial S/\partial A$ (which are the two bifurcation points, and the points $\delta_2 = 13.5°$ and $\delta_2 = 23.0°$ to which we alluded above), we may multiply Eq. (4.1a) by $(\partial S/\partial A)^{-1}$ and substitute the resulting expression for $\delta A_r^{(j+1)}$ into Eq. (4.1b). This gives four real inhomogeneous equations for the *six* unknowns, $\delta p_1^{(j+1)}$, $\delta p_2^{(j+1)}$ and the four $\delta\sigma_{l_k}^{(j+1)}$. Now we write the corresponding equations for the alternative amplitude F' and obtain four more equations for the six unknowns $\delta p_1'^{(j+1)}$, $\delta p_2'^{(j+1)}$, and the same four $\delta\sigma_{l_k}^{(j+1)}$. Evidently we have in all eight equations for eight unknowns, and in general we can find a solution. Hence we generate new amplitudes that satisfy the Crichton requirement $|F + \delta F| = |F' + \delta F'|$. Evidently, since $\delta\sigma_f$ contains only one Legendre coefficient, the one-dimensional degree of freedom corresponds precisely to following the SPD Crichton curve.

It is possible with the same method to obtain nontrivial continuations away from the Crichton SPD cross section by including some new components $\delta\sigma_l$, $l > 4$, in $\delta\sigma_f$. If the number of the new $\delta\sigma_l$ is finite then $\sigma + \delta\sigma$ will still be a polynomial, but $F + \delta F$ and $F' + \delta F'$ will in general possess parasitic branch cuts which can be kept out of the ellipse of integration by choosing $\delta\sigma_f$ small enough. It is possible that, by a sequence of Newton-steps, one could get rid of the branch cuts and finish with new polynomial amplitudes $F + \Delta F$ and $F' + \Delta F'$, of a higher degree than F and F'. For example, there may well exist a continuous connection between the SPD ambiguities and some, or all, of the SPDF cases that have been studied by Berends and Ruijsenaars.

The above demonstrations have been based on the Newton method; but a more satisfactory existence proof may easily be constructed by using the Hildebrandt—Graves theorem, as in Sec. 2; and one could also consider then nonpolynomial changes in σ.

ACKNOWLEDGMENTS

We should like to thank the following people for helpful discussions: C. Itzykson, P. W. Johnson, A. Martin, R. L. Warnock. One of the authors (M. de Roo) has carried out this work as a scientific staff member of the Stichting F. O. M. (Foundation for Fundamental Research on Matter), which is financially supported by the Netherlands Organization for Pure Scientific Research (Z. W. O.).

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